Finite groups in which the real character degrees and real class sizes are prime powers

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Let X be a finite set of positive integers and $\pi(X)$ the set of primes that divide some element in X. The *prime graph* on X is the graph that has $\pi(X)$ as vertices and two elements $p, q \in \pi(X)$ are joined if there is $x \in X$ such that pq divides x. If X is the set of real character degrees or real class sizes, the prime graph on X is denoted respectively $\Delta_r(G)$ and $\Delta_r^*(G)$. Some well known results in group theory have an alternative formulation in terms of prime graphs. For example, the Theorems of Ito-Michler and Thompson can be stated requiring that 2 is an isolated vertex or a complete vertex in $\Delta_r(G)$ and $\Delta_r^*(G)$. This motivated a strong interest on how the shape of prime graphs can impact the structure of the group G.

If $G = H \times K$, then every vertex in the prime graph of H is joined to every vertex in the prime graph of K. This is a typical example of the fact that prime graphs "tend to have many edges"; indeed, it turned out that non-adjacency in the prime graph of G has great impact on the structure of G. In this spirit, we studied the groups G such that $\Delta_r(G)$ has no edges, namely groups in which all real irreducible characters have prime-power degrees. With similar techniques, we studied the groups in which all the sizes of real classes are prime powers, i.e. $\Delta_r^*(G)$ has no edges.