The growth of abelian sections

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Originally motivated by negative results about expansion in Cayley graphs [2], for an arbitrary group G and every $n \geq 1$, we define the **abelianization growth** function

$$ab_n(G) := \sup_{|G:H| \leq n} |H/H'|,$$

which measures how the abelian sections grow with the index of the subgroups. Although at first sight it looks interesting only for infinite groups, the study of $ab_n(G)$ is in truth a matter of *finite* groups. Moreover, it is closely related to another function, the **representation growth** of G, which counts the number of (inequivalent) irreducible representations of dimension at most n.
In this seminar we present some results about $ab_n(G)$, the following two estimates in the particular. The first says that the abelian sections indeed *grow* in all the interesting cases.

Proposition 1 (Proposition 1.2 in [4])**.** *Let G be a finitely generated residually finite group. Then*

$$ab_n(G) \geq \exp\left(\frac{\log n}{6(\log\log n)^2}\right)$$

*for infinitely many n. Moreover, if G has arbitrarily large **solvable** quotients,*

$$ab_n(G) \geq \exp\left(\frac{\log n}{3\log\log n}\right)$$

for infinitely many n.

The second bound is in the opposite direction, and concerns groups with a very slow growth.

Theorem 2 (Theorem 1.3 in [4])**.** *There exists an absolute constant $\alpha > 0$ such that, for every group G without abelian composition factors, one has*

$$ab_n(G) \leq n^{\alpha}$$

for every $n \geq 1$.

Finally, we analyze the abelianization growth of symmetric and alternating groups. This is rather delicate and interesting (as it seems to be quite close to the lower bound of Proposition 1), and leads to collide with a conjecture of Kovács and Praeger [1], which asked for a sharp bound on the abelianization of a transitive group. We settle this conjecture in the affirmative [3].

References

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