Coprime partitions and Jordan totient functions

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Abstract

The study of partitions with a fixed number k of parts satisfying some coprimality condition has revealed to be very fruitful for analysing the normal covering number $\gamma(S_n)$ of the symmetric group S_n, that is, the smallest number of conjugacy classes of proper subgroups needed to cover S_n. In order to efficiently bound $\gamma(S_n)$, it is not necessary to deal with partitions into k parts for every possible $k \leq n$ and the focus is on $k = 2, 3, 4$. Recently, using knowledge about partitions into three parts Bubboloni, Praeger and Spiga [1] have shown that, for n even,

$$\gamma(S_n) \geq \frac{n}{2}\left(1 - \sqrt{1 - 4/\pi^2}\right) - \frac{\sqrt{17}}{2} n^{3/4}.$$

Similar results about S_n for n odd could greatly benefit from knowing more about partitions into four parts, especially those satisfying suitable coprimality conditions. To start with, one should find an exact formula for the number $p'_4(n)$ of coprime partitions of n into four parts. This initial motivation inspired the research presented in this talk.

On the other hand, shedding light on the number $p'_k(n)$ of coprime partitions of n into $k \geq 2$ parts seems to be of recent interest in the scientific community. The aim is representing $p'_k(n)$ as linear combinations of classic number theoretic functions.

Let J_k denote the Jordan totient function of degree $k \geq 0$. Recall that, for every $n \in \mathbb{N}$, $J_k(n) = \sum_{d|n} d^k \mu(n/d)$, where μ is the Möbius function. In [2], it is proved that $p'_3(n) = \frac{J_2(n)}{12}$ holds for $n \geq 4$ and it is also clear that $p'_2(n) = \frac{J_1(n)}{2}$ holds for $n \geq 3$. So, one can ask if similar results could hold for every k. We show that those two situations are pure miracles.

Theorem 1. *$p'_k(n)$ is a $\mathbb{C}$-linear combination of the Jordan totient functions for n sufficiently large if and only if $k \in \{2, 3\}$.*

The feeling is that the class of the Jordan totient functions is too restrictive and some generalizations of them are needed. We consider then three generalizations which are finely linked together: the Jordan root totient functions, the Jordan modulo totient functions and the Jordan-Dirichlet totient functions. We show the following.

Theorem 2. *p'_k is a $\mathbb{C}$-linear combination of the Jordan root totient functions in the entire domain $n \geq 1$.*

We explicitly find the coefficients of such $\mathbb{C}$-linear combination and show how to deduce the expression of the Jordan root totient functions involved making use of the Jordan modulo totient functions and the Jordan-Dirichlet totient functions.

References

[1] D. Bubboloni, C. E. Praeger, P. Spiga, Linear bounds for the normal covering number of the symmetric and alternating groups, *Monatsh. Math.* **191** (2020), 229–247.

[2] M. E. Bachraoui, Relatively prime partitions with two and three parts, *Fibonacci Quart.* **46/47** (2008/09), 341–345. .

[3] D. Bubboloni, F. Luca, Coprime partitions and Jordan totient functions, arXiv:2007.05972v4, to appear on J. Number Theory.