CHARACTERS OF π'-DEGREE

EUGENIO GIANNELLI, A. A. SCHAEFFER FRY, AND CAROLINA VALLEJO RODRÍGUEZ

ABSTRACT. Let G be a finite group and let π be a set of primes. Write $\mathrm{Irr}_{\pi'}(G)$ for the set of irreducible characters of degree not divisible by any prime in π. We show that if π contains at most two prime numbers and the only element in $\mathrm{Irr}_{\pi'}(G)$ is the principal character, then $G = 1$.

INTRODUCTION

Let G be a finite group and let π be a set of primes. Write $\mathrm{Irr}_{\pi'}(G)$ for the set of irreducible characters of degree not divisible by any prime in π. If $\pi = \{p\}$, then we use the standard notation $\mathrm{Irr}_{p'}(G) = \mathrm{Irr}_{\pi'}(G)$. The condition $\mathrm{Irr}_{p'}(G) = \{1_G\}$ implies that $G = 1$ in an elementary way. Indeed, in such situation we have that G is a p'-group, since the order of G is the sum of the squares of the degrees of its irreducible characters. Hence p does not divide the degree of any irreducible character of G and $\mathrm{Irr}(G) = \mathrm{Irr}_{p'}(G) = \{1_G\}$ implies $G = 1$ as wanted. We show that the same result holds if π contains at most two primes.

Theorem A. *Let $\pi = \{p, q\}$ be a set of primes. If $\mathrm{Irr}_{\pi'}(G) = \{1_G\}$, then $G = 1$.*

We remark that the result no longer holds if $|\pi| > 2$. For example, if $\pi = \{2, 3, 5\}$ then, of course, $\mathrm{Irr}_{\pi'}(\mathsf{A}_5) = \{1\}$.

Often in representation theory of finite groups we find a duality between statements on irreducible characters and corresponding ones on conjugacy classes. For instance if p is a prime and the conjugacy class of the identity is the unique conjugacy class of p'-size of G, then $G = 1$. This is the dual statement of the one for irreducible characters described in the first paragraph of this section. We care to remark that the conjugacy class-version of Theorem A does not hold. For instance, the conjugacy class sizes of A_5 are 1, 15, 20, 12 and 12, so for every pair of primes π dividing its order, the identity is the only conjugacy class of A_5 of π'-size.

Our proof of Theorem A relies on the Classification of Finite Simple Groups. We do not know if a CFSG-free proof might exist or if this result heavily depends on properties inherent to the representations of simple groups.

The key observation to prove Theorem A is that for a simple group G and any set $\pi = \{p, q\}$ of primes dividing the order of G, there exists some non-principal character in $\mathrm{Irr}_{\pi'}(G)$. Let $\Gamma'(G)$ be the undirected graph defined as follows. The set of vertices of $\Gamma'(G)$ is the set of primes dividing the order of G, denoted $\pi(|G|)$. Two vertices p and q

are adjacent if there is some $\chi \in \mathrm{Irr}(G)\backslash\mathrm{Lin}(G)$ such that $\chi(1)$ is not divisible by p nor by q. Here $\mathrm{Lin}(G)$ denotes the set of linear characters of G. With this, the claim above can be stated in the following way.

Theorem B. *If G is a non-abelian simple group, then $\Gamma'(G)$ is complete.*

In fact, Theorem A implies that $\Gamma'(G)$ is complete for every perfect group G. Surprisingly enough, if π consists of two primes, then there are many examples where $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$. For instance, this is the case if $G = \mathrm{PSL}_2(27)\cdot C_3$ and $\pi = \{3, 13\}$. Infinitely many other examples of this phenomena can be found among symmetric, general linear, and general unitary groups as shown by Theorems 2.8 and 3.5 below. Further, we describe the structure of $\Gamma'(G)$ in the latter cases in Corollaries 2.9 and 3.6.

Finally, we also analyze the opposite situation: we characterize finite groups G with totally disconnected graph $\Gamma'(G)$.

Theorem C. *Let G be a group. Then $\Gamma'(G)$ is totally disconnected if, and only if, G is solvable and $\mathbf{N}_G(H) \cap G' = H'$ for every π-Hall subgroup H of G, where π is any pair of primes dividing the order of G.*

The paper is structured as follows. In Section 1 we prove Theorem A assuming that Theorem B holds. We also prove Theorem C, using previous results of Bianchi, Chillag, Lewis, and Pacifici [BCLP07] and of Navarro and Wolf [NW02]. The rest of the paper is devoted to the proof of Theorem B on finite simple groups. In Section 2, we prove that $\Gamma'(G)$ is complete whenever G is an alternating group, and we describe $\Gamma'(G)$ for symmetric groups. In Section 3, we prove that $\Gamma'(G)$ is complete when G is a sporadic group or simple group of Lie type, completing the proof of Theorem B by applying the Classification of Finite Simple Groups. We also provide there a description of $\Gamma'(G)$ when G is a general linear or general unitary group.

1. On Theorems A and C

Assuming that Theorem B holds, which follows from Corollary 2.10 and Theorem 3.4 below, we can prove Theorem A.

Proof of Theorem A. By way of contradiction assume that $G > 1$. We may assume that $|\pi| = 2$ and that p and q divide the order of G, otherwise the result follows from the case where $|\pi| = 1$ treated in the introduction. The fact that $\mathrm{Lin}(G) \subseteq \mathrm{Irr}_{\pi'}(G) = \{1_G\}$ forces G to be perfect. Moreover, if $N \lhd G$ has index coprime to p, then $\mathrm{Irr}_{\pi'}(G/N) = \mathrm{Irr}_{q'}(G/N) = \{1_{G/N}\}$ implies $N = G$. Similarly, one concludes that G has no normal subgroup of index coprime to q. If we let $M \lhd G$ be the first (proper) term in a composition series of G, then $S = G/M$ is a simple non-abelian group of order divisible by p and q.

Since the property is inherited by quotients of G, we have that $\text{Irr}_{\pi'}(G/M) = \{1_{G/M}\}$. By Theorem B we conclude that $G = M$, and this is a contradiction. $\square$

The proof of Theorem C relies on [BCLP07] and [NW02]. We will first show that if $\Gamma'(G)$ is totally disconnected, then the group G must be solvable. For a group G, the common-divisor character degree graph $\Gamma(G)$ of G is defined as follows. The vertices of $\Gamma(G)$ are the degrees of the irreducible characters of G, and two vertices a and b are adjacent if $\gcd(a, b) > 1$. In [BCLP07], the authors prove that if $\Gamma(G)$ is connected, then G is solvable.

Lemma 1.1. *Let G be a group with totally disconnected $\Gamma'(G)$. Then G is solvable.*

Proof. By Burnside's $p^a q^b$-theorem we may assume that the order of G is divisible by at least three different primes. Since $\Gamma'(G)$ is totally disconnected, if the order of G is divisible by m primes, then the degree of every non-linear irreducible character of G is divisible by at least $m - 1$ primes. In particular, $\Gamma(G)$ is complete. We conclude that G is solvable by [BCLP07, Main Theorem]. $\square$

The condition $\text{Irr}_{\pi'}(G) = \text{Lin}(G)$ for solvable groups was studied in [NW02].

Theorem 1.2 (Navarro, Wolf). *Let G be a solvable group and let π be any set of primes. Let H be a Hall π-group of G. Then $\text{Irr}_{\pi'}(G) = \text{Lin}(G)$ if, and only if, $\mathbf{N}_G(H) \cap G' = H'$.*

Proof. This is Corollary 3 in [NW02]. $\square$

These results allows us to characterize the groups G with totally disconnected $\Gamma'(G)$.

Proof of Theorem C. If $\Gamma'(G)$ is totally disconnected, then by Lemma 1.1, the group G is solvable and the direct implication follows from Theorem 1.2. The reverse implication follows directly from Theorem 1.2. $\square$

We find interesting to mention that, together with Theorem A, the Navarro-Wolf theorem also provides a necessary condition for a solvable group to have the same set of character degrees as a perfect group (this problem was studied in [N15] and [NR14], where the authors show that such solvable groups do exist).

Corollary 1.3. *Let G be a solvable group. Suppose that G has the same set of character degrees as some perfect group. Then for any pair π of primes dividing the order of G and for every π-Hall subgroup H of G, one has $\mathbf{N}_G(H) \cap G' \neq H'$.*

Proof. Let K be a perfect group such that G and K have the same set of character degrees. As a consequence of Theorem A, $\Gamma'(K)$ is complete, and it is easy to see that also $\Gamma'(G)$ is complete. Then $\text{Irr}_{\pi'}(G) = \text{Lin}(G)$ for every pair of primes π dividing the order of G. The claim follows from Theorem 1.2. $\square$

This necessary condition in Corollary 1.3 is not a sufficient condition for a solvable group G to have the same set of character degrees as a perfect group. For instance, let $G = D_8 \times E_3 \times E_5$, where here E_p denotes an extraspecial group of order p^3. Then the set of character degrees of G is $\{1, 2, 3, 5, 6, 15, 20, 30\}$. Suppose that K is a perfect group with this set of character degrees. Let M be a maximal normal subgroup of K, so that $S = K/M$ is simple non-abelian. Then the set of character degrees of S would be contained in $\{1, 2, 3, 5, 6, 15, 20, 30\}$, a contradiction.

We end this section with the description of $\Gamma'(G)$ for nilpotent groups.

Remark 1.4. Let G be a nilpotent non-abelian group of order $|G| = p_1^{a_1} \cdots p_k^{a_k}$ for primes p_i, $a_i > 0$ and $k \geqslant 3$. Recalling that G is the direct product of its Sylow subgroups, we see that $\Gamma'(G)$ is complete if, and only if, at most $k - 3$ Sylow subgroups of G are abelian. In the case where $k - 2$ Sylow subgroups are abelian, there is an edge connecting every two primes except for the primes corresponding with the non-abelian Sylow subgroups. In the case where all but one Sylow, say the Sylow p_k-subgroup, are abelian, the subgraph of $\Gamma'(G)$ defined by $\{p_1, \ldots, p_{k-1}\}$ is complete and the vertex p_k is isolated.

2. ALTERNATING GROUPS

The aim of this section is to prove Theorem B for alternating groups.

2.1. **Background.** We recall some basic facts in the representation theory of symmetric groups. Standard references for this topic are [Ja79], [JK81] and [Ol94]. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a finite non-increasing sequence of positive integers. If $n = \sum \lambda_i$, then we say that λ is a partition of n and we write $\lambda \vdash n$ or, sometimes, $|\lambda| = n$. We denote by $\mathcal{P}(n)$ the set of partitions of n. With a slight abuse of notation, given a sequence of partitions $T = (\mu_1, \ldots, \mu_t)$ we will write $|T|$ to denote the number $|\mu_1| + \cdots + |\mu_t|$.

The Young diagram of a partition λ is the set

$$[\lambda] = \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leqslant i \leqslant \ell, 1 \leqslant j \leqslant \lambda_i\},$$

where we orient $\mathbb{N} \times \mathbb{N}$ with the x-axis pointing right and the y-axis pointing down. We denote by λ' the conjugate partition of λ, whose Young diagram is obtained from that of λ by a reflection over the main diagonal.

Given $(r, c) \in [\lambda]$, the corresponding hook $H_{(r,c)}(\lambda)$ is the set defined by

$$H_{(r,c)}(\lambda) = \{(r, y) \in [\lambda] \mid y \geqslant c\} \cup \{(x, c) \in [\lambda] \mid x \geqslant r\}.$$

We set $h_{r,c}(\lambda) = |H_{(r,c)}(\lambda)| = 1 + (\lambda_r - c) + (\lambda'_c - r)$. We refer to $h_{r,c}(\lambda)$ as the hook-length of $H_{(r,c)}(\lambda)$. We denote by $\mathcal{H}(\lambda)$ the multiset of hook-lengths in $[\lambda]$. For $e \in \mathbb{N}$ we let $\mathcal{H}^e(\lambda) = \{(r, c) \in [\lambda] \mid e \text{ divides } h_{r,c}(\lambda)\}$. If $(r, c) \in \mathcal{H}^e(\lambda)$, then we say that $H_{(r,c)}(\lambda)$ is an e-hook of λ, so that $|\mathcal{H}^e(\lambda)|$ is the number of e-hooks of λ. We record here an elementary observation (see [Ol94, Corollary 1.7]) that will be quite useful later in this section.

Lemma 2.2. *Let* $e, f \in \mathbb{N}$ *and suppose that* $h_{r,c}(\lambda) = ef$. *Then* $|\mathcal{H}^e(\lambda) \cap H_{(r,c)}(\lambda)| = f$.

We denote by $\lambda - H_{(r,c)}(\lambda)$ the partition obtained by removing the e-hook $H_{(r,c)}(\lambda)$ from λ (see [Ol94, Chapter I] for the precise definition of this process). The e-core $C_e(\lambda)$ of λ is the partition obtained from λ by successively removing all e-hooks. The e-quotient $Q_e(\lambda) = (\lambda^{(0)}, \ldots, \lambda^{(e-1)})$ is another important combinatorial object, defined for instance in [Ol94, Section 3]. The number of e-hooks to be removed from λ to obtain $C_e(\lambda)$ is called the e-weight $w_e(\lambda)$. By [Ol94, 3.6] we derive the following equations.

$$(2.1) \quad |\lambda| = ew_e(\lambda) + |C_e(\lambda)|, \text{ and } w_e(\lambda) = |\mathcal{H}^e(\lambda)| = |Q_e(\lambda)| = |\lambda^{(0)}| + \cdots + |\lambda^{(e-1)}|.$$

Let $T_0^Q(\lambda) = (\lambda)$, $T_1^Q(\lambda) = Q_e(\lambda) = (\lambda^{(0)}, \ldots, \lambda^{(e-1)})$ and for $k \geqslant 1$ we define $T_{k+1}^Q(\lambda)$ to be the sequence of e^{k+1} partitions given by

$$T_{k+1}^Q(\lambda) = ((\lambda^{(i_1, \ldots, i_k)})^{(0)}, \ldots, (\lambda^{(i_1, \ldots, i_k)})^{(e-1)}),$$

where $(i_1, \ldots, i_k) \in \{0, 1, \ldots, e - 1\}^k$. The collection of all the sequences $T_j^Q(\lambda)$ for $j \geqslant 0$ is known as the *e-quotient tower* of λ. It is not too difficult to see that $|Q_{e^k}(\lambda)| = |T_k^Q(\lambda)|$, for all $k \in \mathbb{N}$. If $T_k^Q(\lambda) = (\mu_1, \ldots, \mu_{e^k})$ then we let $T_k^C(\lambda) = (C_e(\mu_1), \ldots, C_e(\mu_{e^k}))$. The collection of all the sequences $T_j^C(\lambda)$ for $j \geqslant 0$ is known as the *e-core tower* of λ.

Let now p be a prime. As shown in [Ol94, Chap. II], every partition of a given natural number is uniquely determined by its p-core tower. Using the definitions given above we observe that for all $k \in \mathbb{N}_0$ we have that

$$(2.2) \qquad |\mathcal{H}^{p^k}(\lambda)| = |T_k^Q(\lambda)| = \sum_{j \geqslant k} |T_j^C(\lambda)| p^{j-k}, \quad \text{in particular} \quad |\lambda| = \sum_{j \geqslant 0} |T_j^C(\lambda)| p^j.$$

Partitions of n correspond canonically to the irreducible characters of S_n. We denote by χ^λ the irreducible character naturally labelled by $\lambda \vdash n$. We use the notation $\lambda \vdash_{p'} n$ to say that $\chi^\lambda \in \mathrm{Irr}_{p'}(\mathsf{S}_n)$. We recall that $(\chi^\lambda)_{\mathsf{A}_n}$ is irreducible if, and only if, $\lambda \neq \lambda'$. Otherwise $(\chi^\lambda)_{\mathsf{A}_n} = \varphi + \varphi^g$ for some $\varphi \in \mathrm{Irr}(\mathsf{A}_n)$ and $g \in \mathsf{S}_n \smallsetminus \mathsf{A}_n$. (See [JK81, Thm. 2.5.7].) The following result was first proved by MacDonald [Mac71] and it is crucial for our purposes.

Theorem 2.3. *Let p be a prime and let n be a natural number with p-adic expansion $n = \sum_{j=0}^k a_j p^j$. Let λ be a partition of n. Then*

$$\nu_p(\chi^\lambda(1)) = \left(\sum_{j \geqslant 0} |T_j^C(\lambda)| - \sum_{j=0}^k a_j \right) / (p - 1).$$

Moreover, $\nu_p(\chi^\lambda(1)) = 0$ if, and only if, $|T_j^C(\lambda)| = a_j$ for all $j \in \mathbb{N}_0$.

Here for a natural number m, we denoted by $\nu_p(m)$ the maximal integer k such that p^k divides m. We will keep this notation for the rest of the article.

A useful consequence of Theorem 2.3 is stated in the following lemma. This is well-known to experts in the field. For the reader's convenience, we only include a brief proof.

Lemma 2.4. *Let p, n and λ be as in Theorem 2.3. Then $\lambda \vdash_{p'} n$ if and only if $|\mathcal{H}^{p^k}(\lambda)| = a_k$ and $C_{p^k}(\lambda) \vdash_{p'} n - a_k p^k$.*

Proof. Assume that $|\mathcal{H}^{p^k}(\lambda)| = a_k$ and that $C_{p^k}(\lambda) \vdash_{p'} n - a_k p^k$. Since $p^{k+1} > n$, it follows that $|T_s^C(\lambda)| = 0$ for all $s > k$. Hence using equation (2.2) above, we get that $a_k = |\mathcal{H}^{p^k}(\lambda)| = |T_k^C(\lambda)|$. By [Ol94, Theorem 3.3] one deduces that $T_j^C(C_{p^k}(\lambda)) = T_j^C(\lambda)$ for all $0 \leqslant j < k$. Using Theorem 2.3, we deduce that $\lambda \vdash_{p'} n$. The converse implication is a direct consequence of Theorem 2.3. $\qquad \square$

Let $\mathcal{L}(n) := \{(n - x, 1^x) \mid 0 \leqslant x \leqslant n - 1\}$ be the set all the *hook partitions* of the natural number n. The following fact is an immediate consequence of Lemma 2.4.

Lemma 2.5. *Let $k \in \mathbb{N}$ and let p be a prime. Then $\chi^\lambda \in \mathrm{Irr}_{p'}(\mathsf{S}_{p^k})$ if and only if $\lambda \in \mathcal{L}(p^k)$.*

We conclude our preliminaries with the following technical lemma.

Lemma 2.6. *Let $k \in \mathbb{N}_{>0}$ and let $\varepsilon \in \{0, 1\}$. Let $n = 2^k + \varepsilon$ and let $\lambda \vdash n$. If $\mathcal{H}^{2^k}(\lambda) = \varnothing$ and $|\mathcal{H}^{2^{k-1}}(\lambda)| = 2$ then $\nu_2(\chi^\lambda(1)) = 1$.*

Proof. By equations (2.1) we have that $n = |C_{2^{k-1}}(\lambda)| + 2^{k-1}|\mathcal{H}^{2^{k-1}}(\lambda)|$. It follows that $|C_{2^{k-1}}(\lambda)| = \varepsilon$. In particular $C_{2^{k-1}}(\lambda)$ is a 2-core partition. Since removing a 2^{k-1}-hook does not change the 2-core of a partition, we have that $C_2(\lambda) = C_2(C_{2^{k-1}}(\lambda)) \vdash \varepsilon$. Thus $|T_0^C(\lambda)| = |C_2(\lambda)| = \varepsilon$. Moreover, using equations (2.2), we see that $|T_j^C(\lambda)| = 0$ for all $j \geqslant k$. Hence, again by equations (2.2) we conclude that

$$|T_{k-1}^C(\lambda)| = |T_{k-1}^Q(\lambda)| = |\mathcal{H}^{2^{k-1}}(\lambda)| = 2, \text{ and hence } |T_j^C(\lambda)| = 0 \text{ for all } 1 \leqslant j \leqslant k-2.$$

It follows that $\sum_{j \geqslant 0} |T_j^C(\lambda)| = 2 + \varepsilon$, while $\sum_{j=0}^k a_j = 1 + \varepsilon$. The statement now follows from Theorem 2.3. □

2.7. **Main Results.** We are now ready to prove Theorem B for alternating groups. In order to do this we will first completely describe $\Gamma'(\mathsf{S}_n)$.

Theorem 2.8. *Let $n \geqslant 5$ and let q and p be distinct primes such that $2 \leqslant q, p \leqslant n$. Let $\pi = \{p, q\}$. Then $\mathrm{Irr}_{\pi'}(\mathsf{S}_n) = \mathrm{Lin}(\mathsf{S}_n)$ if and only if $q = 2$ and*

$$n = \begin{cases} 2^k = p^m + 1 & \text{or,} \\ 2^k + 1 = p^m. \end{cases}$$

Proof. We aim to construct a partition $\lambda \in \mathcal{P}(n) \smallsetminus \{(n), (1^n)\}$ such that $\chi^\lambda \in \mathrm{Irr}_{\pi'}(\mathsf{S}_n)$. Let

$$\sum_{i=1}^t a_i p^{m_i} = n = \sum_{i=1}^r b_i q^{k_i},$$

be the p-adic and respectively the q-adic expansions of n, where $m_1 > m_2 > \cdots m_t \geqslant 0$ and $k_1 > k_2 > \cdots k_r \geqslant 0$. It is clear that $b_1 q^{k_1} \neq a_1 p^{m_1}$.

Suppose that $b_1 q^{k_1} < a_1 p^{m_1}$ (the statement in the opposite setting will be proved just by swapping p with q, a_1 with b_1 and m_1 with k_1 in what follows). Consider the partition λ of n defined by:

$$\lambda = (n - b_1 q^{k_1}, n - a_1 p^{m_1} + 1, 1^{b_1 q^{k_1} - (n - a_1 p^{n_1} + 1)}).$$

Observe that λ is a well-defined partition as $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_j = 1$, for all $j \geqslant 3$. Since $h_{1,1}(\lambda) = a_1 p^{m_1}$, we deduce that $|\mathcal{H}^{p^{m_1}}(\lambda)| \geqslant a_1$ from Lemma 2.2. By equations (2.1) we know that $|\mathcal{H}^{p^{m_1}}(\lambda)| = w_{p^{m_1}}(\lambda) \leqslant a_1$. It follows that $|\mathcal{H}^{p^{m_1}}(\lambda)| = a_1$. Moreover

$$C_{p^{m_1}}(\lambda) = \lambda - H_{1,1}(\lambda) = (n - a_1 p^{m_1}) \vdash_{p'} n - a_1 p^{m_1}.$$

Using Lemma 2.4, we conclude that $\chi^\lambda \in \mathrm{Irr}_{p'}(\mathsf{S}_n)$. On the other hand, using a similar argument, we can also show that $\chi^\lambda \in \mathrm{Irr}_{q'}(\mathsf{S}_n)$. This follows from Lemma 2.4 because $h_{2,1}(\lambda) = b_1 q^{k_1}$, hence $|\mathcal{H}^{q^{k_1}}(\lambda)| = w_{q^{k_1}}(\lambda) = b_1$, and again it is routine to check that $C_{q^{k_1}}(\lambda) = (n - b_1 q^{k_1}) \vdash_{q'} n - b_1 q^{k_1}$. We obtain that $\chi^\lambda \in \mathrm{Irr}_{\pi'}(\mathsf{S}_n)$. To conclude we need to make sure that $\chi^\lambda \notin \mathrm{Lin}(\mathsf{S}_n)$ (i.e. $\lambda \notin \{(n), (1^n)\}$).

Since $\lambda_2 \geqslant 1$ we notice that $\lambda \in \{(n), (1^n)\}$ if and only if $\lambda = (1^n)$ and this happens if and only if $n - a_1 p^{m_1} = \lambda_2 - 1 = 0$ and $n - b_1 q^{k_1} = \lambda_1 = 1$. Equivalently, $\lambda \in \{(n), (1^n)\}$ if and only if

$$n = ap^m = bq^k + 1.$$

(To ease the notation we renamed a_1, m_1, b_1 and k_1 by a, m, b and k respectively.) In this very specific situation $\lambda = (1^n)$, hence we need to pick a different partition. In order to make this new choice we distinguish two main cases, depending on $2 \in \{p, q\}$ or not.

Let us first assume that $2 \notin \{p, q\}$. If $b \geqslant 2$ then let $\mu \vdash n$ be defined as follows:

$$\mu = (1 + (b - 1)q^k, 1^{q^k}).$$

Notice that $h_{1,1}(\mu) = ap^m$, $h_{1,2}(\mu) = (b - 1)q^k$ and $h_{2,1}(\mu) = q^k$. Hence from Lemma 2.2 we obtain that $|\mathcal{H}^{p^m}(\mu)| = a$ and that $|\mathcal{H}^{q^k}(\mu)| = b$. Moreover, $C_{p^m}(\mu) = \varnothing$ and $C_{q^k}(\mu) = (1)$. Using Lemma 2.4 we deduce that $\chi^\mu \in \mathrm{Irr}_{\pi'}(\mathsf{S}_n)$. Otherwise if $b = 1$, then $a = 2c$ is even and we let

$$\mu = (cp^m, 2, 1^{cp^m - 2}).$$

It is now routine to check that $h_{1,1}(\mu) = q^k$, $h_{1,2}(\mu) = h_{2,1}(\mu) = cp^m$, $|\mathcal{H}^{p^m}(\mu)| = a$, $|\mathcal{H}^{q^k}(\mu)| = 1$, $C_{p^m}(\mu) = \varnothing$ and $C_{q^k}(\mu) = (1)$. Lemma 2.4 implies that $\chi^\mu \in \mathrm{Irr}_{\pi'}(\mathsf{S}_n)$.

To conclude we now analyze the case where $2 \in \{q, p\}$. We have two substantially different cases to consider. Namely $n = ap^m = 2^k + 1$ and $n = ap^m + 1 = 2^k$.

- If $n = 2^k = ap^m + 1$ then $\chi^\mu \in \mathrm{Irr}_{2'}(\mathsf{S}_n)$ if and only if $\mu \in \mathcal{L}(n)$, by Lemma 2.5. If $a > 1$ then we let μ be the partition of n defined as follows.

$$\mu = (1 + (a - 1)p^m, 1^{p^m}) \in \mathcal{L}(n) \smallsetminus \{(n), (1^n)\}.$$

We observe that $h_{1,2}(\mu) = (a - 1)p^m$ and $h_{2,1}(\mu) = p^m$. Hence $|\mathcal{H}^{p^m}(\mu)| = a$ by Lemma 2.2, and $C_{p^m}(\mu) = (1)$. Therefore $\chi^\mu \in \mathrm{Irr}_{\{2,p\}'}(\mathsf{S}_n)$ by Lemma 2.4.

On the other hand, if $a = 1$ then it is not difficult to see that the only characters in $\mathrm{Irr}_{p'}(\mathsf{S}_n)$ labelled by hook partitions are the trivial and the sign character. More precisely, we have that $\mathrm{Irr}_{p'}(\mathsf{S}_n) \smallsetminus \mathrm{Lin}(\mathsf{S}_n) = \{(n - x, 2, 1^{x-2}) \mid 2 \leqslant x \leqslant n - 2\}$. Hence $\mathrm{Irr}_{\{2,p\}'}(\mathsf{S}_n) = \mathrm{Lin}(\mathsf{S}_n)$.

- If $n = 2^k + 1 = ap^m$ and $a > 1$ then we define $\mu \vdash n$ as follows.

$$\mu = (1 + (a - 1)p^m, 2, 1^{p^m - 2}).$$

It is routine to check that $h_{11}(\mu) = 2^k$, $h_{1,2}(\mu) = (a - 1)p^m$ and that $h_{2,1}(\mu) = p^m$. We conclude that $\chi^\mu \in \mathrm{Irr}_{\{2,p\}'}(\mathsf{S}_n)$, by Lemma 2.4.

If $a = 1$ then $\chi^\mu \in \mathrm{Irr}_{p'}(\mathsf{S}_n)$ if and only if $\mu \in \mathcal{L}(n)$, by Lemma 2.5. It is now easy to check that the only characters in $\mathrm{Irr}_{2'}(\mathsf{S}_n)$ labelled by hook partitions are the trivial and the sign character. Hence $\mathrm{Irr}_{\{2,p\}'}(\mathsf{S}_n) = \mathrm{Lin}(\mathsf{S}_n)$. $\square$

Theorem 2.8 can be reformulated as follows.

Corollary 2.9. *The graph $\Gamma'(\mathsf{S}_n)$ is not complete if and only if $n = 2^k = p^m + 1$ or $n = 2^k + 1 = p^m$. In these cases the subgraphs of $\Gamma'(\mathsf{S}_n)$ defined by $\pi(|\mathsf{S}_n|) \smallsetminus \{2\}$ and by $\pi(|\mathsf{S}_n|) \smallsetminus \{p\}$ are complete, and there is no edge between 2 and p.*

Corollary 2.10. *Let $p \neq q$ be two primes and define $\pi := \{p, q\}$. Then $|\mathrm{Irr}_{\pi'}(\mathsf{A}_n)| > 1$, for all $n \geqslant 5$.*

Proof. By Theorem 2.8 there exists $\lambda \in \mathcal{P}(n) \smallsetminus \{(n), (1^n)\}$ such that $\chi^\lambda \in \mathrm{Irr}_{\pi'}(\mathsf{S}_n)$, unless $q = 2$ and $n = 2^k = p^m + 1$ or $n = 2^k + 1 = p^m$. In all these cases, let $\varphi \in \mathrm{Irr}(\mathsf{A}_n)$ be an irreducible constituent of $(\chi^\lambda)_{\mathsf{A}_n}$. Then $\varphi(1) \in \{\chi^\lambda(1), \chi^\lambda(1)/2\}$. In particular $\varphi \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$.

Suppose now that $q = 2$ and $n = 2^k = p^m + 1$ or $n = 2^k + 1 = p^m$. We let μ be the partition of n defined as follows:

$$\mu = \begin{cases} (2^{k-1} + 1, 1^{2^{k-1}}) & \text{if } n = p^m = 2^k + 1, \\ \\ (2^{k-1}, 2, 1^{2^{k-1}-2}) & \text{if } n = p^m + 1 = 2^k. \end{cases}$$

(Notice that μ is well defined because $n > 4$.) In both cases we see that $\mathcal{H}^{2^k}(\mu) = \varnothing$ and that $|\mathcal{H}^{2^{k-1}}(\mu)| = 2$. Therefore by Lemma 2.6 we deduce that $\nu_2(\chi^\mu(1)) = 1$. Since $\mu = \mu'$ we conclude that $\varphi(1) = \chi^\mu(1)/2$ is odd, for any irreducible constituent φ of $(\chi^\mu)_{\mathsf{A}_n}$. Moreover, in both cases $h_{1,1}(\mu) = p^m$ and

$$C_{p^m}(\mu) = \begin{cases} \varnothing & \text{if } n = p^m = 2^k + 1, \\ (1) & \text{if } n = p^m + 1 = 2^k. \end{cases}$$

Hence Lemma 2.4 guarantees that $\chi^\mu \in \mathrm{Irr}_{p'}(\mathsf{S}_n)$. We conclude that $\varphi \in \mathrm{Irr}_{\pi'}(\mathsf{A}_n)$. □

3. Groups of Lie Type and Sporadic Groups

The aim of this section is to complete the proof of Theorem B. We begin with the case of the sporadic groups and certain groups of Lie type that may be treated computationally.

Lemma 3.1. *The simple groups $G_2(3)$, $G_2(4)$, $^2G_2(3)'$, $^2F_4(2)'$, and $^2E_6(2)$ and the 26 sporadic groups satisfy Theorem B.*

Proof. This can be seen using GAP and the Character Table Library. □

Let $G = \mathbf{G}^F$ be the group of fixed points of a connected reductive algebraic group $\mathbf{G}$ defined over $\overline{\mathbb{F}}_r$ under a Steinberg map F. Here r is a prime and $\overline{\mathbb{F}}_r$ is an algebraic closure of the finite field of cardinality r. We will call a group G of this form a finite reductive group. Further, let $G^* = (\mathbf{G}^*)^{F^*}$, where $(\mathbf{G}^*, F^*)$ is dual to $(\mathbf{G}, F)$.

The set of irreducible characters $\mathrm{Irr}(G)$ can be written as a disjoint union $\bigsqcup \mathcal{E}(G, s)$ of rational Lusztig series corresponding to G^*-conjugacy classes of semisimple elements $s \in G^*$. The characters in the series $\mathcal{E}(G, 1)$ are called unipotent characters, and there is a bijection $\mathcal{E}(G, s) \to \mathcal{E}(C_{G^*}(s), 1)$. Hence, characters of $\mathrm{Irr}(G)$ may be indexed by pairs (s, ψ), where $s \in G^*$ is a semisimple element, up to G^*-conjugacy, and $\psi \in \mathrm{Irr}(C_{G^*}(s))$ is a unipotent character.

Further, if $\chi \in \mathrm{Irr}(G)$ is indexed by the pair (s, ψ), then the degrees of χ and ψ are related in the following way:

$$(3.1) \qquad\qquad \chi(1) = |G^* : C_{G^*}(s)|_{r'}\psi(1)$$

(see [DM91, Remark 13.24]). Here for a natural number n and a prime p, we denote by $n_{p'}$ the largest integer m dividing n such that $\gcd(m, p) = 1$. Similarly, we will denote the number $p^{\nu_p(n)}$ by n_p.

Lemma 3.2. *Let $G = \mathbf{G}^F$ be a finite reductive group defined over a field of characteristic r. Let $p \neq q$ be two primes and define $\pi := \{p, q\}$. Then $|\mathrm{Irr}_{\pi'}(G)| > 1$.*

Proof. First, suppose $p \neq r$ and $q \neq r$. Then the degree of the Steinberg character St_G is a power of r (see for example [DM91, Corollary 9.3]), and is therefore an element in $\mathrm{Irr}_{\pi'}(G)$. Then we may assume that p is the defining characteristic for G. That is, we assume $r = p$.

Now, let Q be a Sylow q-subgroup of G^* and let $s \in Z(Q)$ be nontrivial. Then s is semisimple, since $q \neq p$, and certainly $Q \leqslant C_{G^*}(s)$. Hence taking χ to be indexed by $(s, 1_{C_{G^*}(s)})$, we have $q \nmid \chi(1)$ and $p \nmid \chi(1)$ from (3.1), so $\chi \in \mathrm{Irr}_{\pi'}(G)$. □

Before we extend the above result to prove Theorem B in the case of simple groups of Lie type, we note the following straightforward but useful lemma.

Lemma 3.3. *Let G be a perfect group and let q be prime. Suppose that $|Z(G)|$ is a power of q and that $\chi \in \mathrm{Irr}(G)$ has degree prime to q. Then $Z(G)$ is in the kernel of χ.*

Proof. Write $Z = Z(G)$. The order $o(\lambda)$ of $\lambda \in \mathrm{Irr}(Z)$ lying under χ must divide $\chi(1)$, since $1_Z = \det(\chi)_Z = \lambda^{\chi(1)}$. □

We are now ready to complete the proof of Theorem B.

Theorem 3.4. *Let S be a finite simple group such that $S = G/Z(G)$ for $G = \mathbf{G}^F$ a finite reductive group of simply connected type defined over $\mathbb{F}_{r^a}$ for some prime r. Let $p \neq q$ be two primes and define $\pi := \{p, q\}$. Then $|\mathrm{Irr}_{\pi'}(S)| > 1$.*

Proof. By Lemma 3.1, we may assume S is not one of the groups listed there, and by Section 2, we may assume S is not isomorphic to an alternating group. We wish to show that the character χ constructed in the proof of Lemma 3.2 can be chosen to be trivial on $Z(G)$. This is satisfied for St_G, so we again assume $r = p$. By [NT13, Lemma 4.4(ii)], it therefore suffices to show that the semisimple element s used in the proof of Lemma 3.2 can be chosen to be contained in $(G^*)'$.

Here G is quasisimple and $|Z(G)| = |G^* : (G^*)'|$, where $(G^*)'$ denotes the commutator subgroup of G^*. Now, if $q \nmid |Z(G)|$, then $Q \leqslant (G^*)'$, where Q is any Sylow q-subgroup of G^*, and the s chosen in Lemma 3.2 is in this case contained in $(G^*)'$. Combining this with Lemma 3.3, we may therefore assume that q divides $|Z(G)|$ but that $|Z(G)|$ is not a power of q. This leaves only the case that $\mathbf{G}$ is of type A.

Hence, for the remainder of the proof, we assume that $G = \mathrm{SL}_n^\epsilon(p^a)$ with $\epsilon \in \{\pm 1\}$. Here $\epsilon = 1$ means $G = \mathrm{SL}_n(p^a)$, $\epsilon = -1$ means $G = \mathrm{SU}_n(p^a)$, and $|Z(G)| = \gcd(n, p^a - \epsilon)$. Recall from above that we may assume n is not a power of q. Writing $\widetilde{G} = \mathrm{GL}_n^\epsilon(p^a)$, we further have

$$G^* \cong \widetilde{G}/Z(\widetilde{G}) \text{ and } (G^*)' \cong GZ(\widetilde{G})/Z(\widetilde{G}) \cong S,$$

and we will make these identifications.

Let $\widetilde{Q}$ be a Sylow q-subgroup of $\widetilde{G}$, so that $Q := \widetilde{Q}Z(\widetilde{G})/Z(\widetilde{G})$ is a Sylow q-subgroup of G^*. Now, if $Z(Q) \cap (G^*)' \neq 1$, we can take s to be a nontrivial element of this intersection, and we are done. So we may assume $Z(Q) \cap (G^*)' = 1$, in which case $Z(Q) \cong Z(Q)(G^*)'/(G^*)' \leqslant G^*/(G^*)'$ and we see that $|Z(Q)|$ divides $\gcd(n, p^a - \epsilon)_q$, since $|G^* : (G^*)'| = \gcd(n, p^a - \epsilon)$, and hence $|Z(\widetilde{Q})|$ divides $(p^a - \epsilon)_q \cdot \gcd(n, p^a - \epsilon)_q$.

Now, writing $n = a_0 + a_1 q + a_2 q^2 + \cdots + a_t q^t$ with $0 \leqslant a_i < q$ for the q-adic expansion of n, we see by [We95] and [CF64] that $\widetilde{Q}$ can be chosen as the direct product

$$\widetilde{Q} = \prod_{i=0}^{t} Q_i^{a_i}$$

where Q_i is a Sylow q-subgroup of $\mathrm{GL}_{q^i}^{\epsilon}(p^a)$, and this identification can be made by embedding the matrices block-diagonally into $\widetilde{G}$. Note that $Z(Q_i)$ contains the Sylow q-subgroup of $Z(\mathrm{GL}_{q^i}^{\epsilon}(p^a)) \cong C_{p^a - \epsilon}$. Then $|Z(\widetilde{Q})| > (p^a - \epsilon)_q^2$, a contradiction, unless n is of the form $q^i + q^j$ for some $0 \leqslant i \leqslant j$. (Note that this includes the case $n = 2q^i$ for some $i \geqslant 1$, which can only occur if $q \neq 2$.) Further, $|Z(\widetilde{Q})| > (p^a - \epsilon)_q \cdot \gcd(n, p^a - \epsilon)_q$ unless $(p^a - \epsilon)_q \mid n_q$. Since $n_q = q^i$, this means we must have $(p^a - \epsilon)_q \mid q^i$.

Now let μ be any nontrivial element in the Sylow q-subgroup of $C_{p^a - \epsilon}$, viewed inside $\mathbb{F}_{p^a}^{\times}$ or $\mathbb{F}_{p^{2a}}^{\times}$, corresponding to the cases $\epsilon = 1$ and -1, respectively. Consider the element $x = (\mu I_{q^i}, I_{q^j}) \in Z(\mathrm{GL}_{q^i}^{\epsilon}(p^a)) \times Z(\mathrm{GL}_{q^j}^{\epsilon}(p^a)) \leqslant Z(\widetilde{Q})$. Then x is a non-central semisimple element of $\widetilde{G}$ in $Z(\widetilde{Q})$ and satisfies $\det(x) = \mu^{q^i} = 1$, where the last equality follows from the fact that $(p^a - \epsilon)_q \mid q^i$. Hence $x \in G$ and therefore the element $s := x Z(\widetilde{G})$ is a nontrivial element of $G Z(\widetilde{G})/Z(\widetilde{G}) = (G^*)'$ in $Z(Q)$, again a contradiction. $\qquad\square$

Using the construction in the proof of Theorem 3.4, we find an analogue of Theorem 2.8 and Corollary 2.9 for the groups $\mathrm{GL}_n^{\epsilon}(r^a)$.

Theorem 3.5. *Let $q \neq p$ be two prime numbers and write $\pi := \{p, q\}$. Let $G = \mathrm{GL}_n^{\epsilon}(r^a)$ for a prime r and $\epsilon \in \{\pm 1\}$. Then $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$ if and only if there is some $k \geqslant 0$ such that $(r, n) = (p, q^k)$ and $q \mid (p^a - \epsilon)$, up to reordering p and q.*

Proof. Note that in this case, we may identify $G \cong G^*$. First assume that $\mathrm{Irr}_{\pi'}(G) = \mathrm{Lin}(G)$. Since the Steinberg character of G has degree $r^{an(n-1)/2}$, it must be that $r = p$ or $r = q$. Without loss, we assume that $r = p$.

Let Q be a Sylow q-subgroup of G. Then any character of degree prime to q can be indexed by a pair (s, ψ), where s centralizes Q and ψ has degree prime to q, using (3.1). However, $\mathrm{Lin}(G)$ is comprised of characters indexed by pairs $(z, 1_G)$ for $z \in Z(G)$ (see [DM91, Proposition 13.30]), so it follows that there are no non-central semisimple elements of G centralizing Q. In particular, this shows that $q \mid (p^a - \epsilon)$, since otherwise $q \nmid |Z(G)|$, so $Q \cap Z(G) = 1$, yielding non-central semisimple elements of G contained in $Z(Q)$. Further, arguing as in the proof of Proposition 3.4, we see $n = q^k$ for some integer k, since otherwise $|Z(Q)| \geqslant (p^a - \epsilon)_q^2$ and there is some semisimple element in $Z(Q)$ not contained in $Z(G)$.

Conversely, assume that $r = p$, $n = q^k$, and $q \mid (p^a - \epsilon)$. Since the centralizer of a semisimple element $s \in G$ is a product of groups of the form $\mathrm{GL}_{n_i}^{\epsilon_i}(p^{a_i})$, [Ma07, Theorem 6.8] and [DM91, Proposition 13.20] yield that the only unipotent characters of $C_G(s)$ with degree prime to p are linear. It therefore suffices to show that if $s \in G$ is semisimple and $Q \leqslant C_G(s)$, then $s \in Z(G)$.

Let $s \in G$ be a semisimple element centralizing Q. For $i \geqslant 0$, let Q_i and T_i denote a Sylow q-subgroup of $\mathrm{GL}_{q^i}^{\epsilon}(p^a)$ and S_{q^i}, respectively. Using [CF64] and [We95], we have $Q = Q_k$ is of the form $Q_1 \wr T_{k-1}$.

Suppose q is odd. If $\epsilon = 1$, then further $Q = Q_0 \wr T_k \cong C_{(p^a-1)_q} \wr T_k$. In particular, in this case, the normal subgroup $C_{(p^a-1)_q}^{q^k}$ of Q may be viewed as the Sylow q-subgroup of a maximally split torus, consisting of diagonal matrices. Since s must centralize this subgroup, we have s is also a diagonal matrix. Further, with this identification, T_k acts via permuting the copies of $C_{(p^a-1)_q}$, and hence the diagonal entries of a diagonal matrix. Then since s must commute with this subgroup, we see that s is a scalar matrix, so is in $Z(G)$.

Now let $\epsilon = -1$, so $G = \mathrm{GU}_{q^k}(p^a) \leqslant \mathrm{GL}_{q^k}(p^{2a})$. Then since $q \mid (p^a + 1)$, [We95, 4(ii)] yields that Q is a Sylow q-subgroup of $\mathrm{GL}_{q^k}(p^{2a})$. As $q \mid (p^{2a} - 1)$, the previous paragraph shows that s must be an element of $Z(G)$.

Finally, assume $q = 2$ and $\epsilon \in \{\pm 1\}$. Then by [CF64], we have $N_G(Q) \cong Q \times C_{(p^a-\epsilon)_{2'}}$, where the factor $C_{(p^a-\epsilon)_{2'}}$ is embedded naturally as the largest odd-order subgroup of $Z(G)$, which shows that an element of $C_G(Q)$ must be a member of $Z(Q)Z(G)$. But since $x \in Z(Q)$ must commute with the action of T_{k-1}, $Z(Q)$ must be comprised of elements of $Q_1^{2^{k-1}}$ whose components are all the same. Further, these components must be in $Z(Q_1)$. Considering the description in [CF64] of the Sylow 2-subgroups Q_1 of $\mathrm{GL}_2^\epsilon(p^a)$, we see that $Z(Q)$ therefore consists of scalar matrices, and hence elements of $Z(G)$. $\qquad\square$

Corollary 3.6. *Let $q \neq p$ be two prime numbers and write $\pi := \{p, q\}$. Let $G = \mathrm{GL}_n^\epsilon(r^a)$ for a prime r and $\epsilon \in \{\pm 1\}$. Then $\Gamma'(G)$ is not complete if and only if there is some $k \geqslant 0$ such that $(r, n) = (p, q^k)$ and $q \mid (p^a - \epsilon)$, up to reordering p and q. In this case, the subgraphs of $\Gamma'(G)$ defined by $\pi(|G|) \smallsetminus \{p\}$ and by $\pi(|G|) \smallsetminus \{q\}$ are complete, and there is no edge between p and q.*

References

[BCLP07] M. Bianchi, D. Chillag, M. Lewis, E. Pacifici. Character degree graphs that are complete graphs. *Proc. Amer. Math. Society*, **135**, No. 3 (2007) 671–676.

[CF64] R. Carter, P. Fong. The Sylow 2-Subgroups of the Finite Classical Groups. *J. Algebra*, 1 (1964), 139–151.

[GAP] The GAP Group, GAP Groups, Algorithms, and Programming, Version 4.4; 2004, http://www.gap-system.org.

[DM91] F. Digne, J. Michel. Representations of finite groups of Lie type. *London Mathematical Society Student Texts*, Vol. 21, Cambridge University Press, Cambridge, 1991.

[Ja79] G. D. James, *The Representation Theory of the Symmetric Groups.* Lecture Notes in Mathematics, vol. 682, Springer, Berlin, 1978.

[JK81] G. James, A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley Publishing Co., Reading, Mass., 1981.

[Mac71] I. G. Macdonald, On the degrees of the irreducible representations of symmetric groups. *Bull. London Math. Soc.*, **3** (1971), 189–192.

[Ma07] Gunter Malle, Height 0 Characters of Finite Groups of Lie Type. *Represent. Theory*, **11** (2007), 192–220.

[N15] G. Navarro. The set of character degrees of a finite group does not determine its solvability. *Proc. Amer. Math. Society*, **143**, No. 3 (2015), 989–990.

[NR14] G. Navarro, N. Rizo. Nilpotent and perfect groups with the same set of character degrees. *J. Algebra Appl.*, **13** (2014), No. 8, 1450061, 3pp.

[NT13] G. Navarro, P. H Tiep, Characters of relative p'-degree over normal subgroups. *Ann. of Math. (2)*, Vol. 178, No. 3 (2013), 1135–1171.

[NW02] G. Navarro, T. Wolf, Variations on Thompon's character degree theorem. *Glasgow Math. J.*, **44** (2002), 371–374.

[Ol94] J. B. Olsson, *Combinatorics and Representations of Finite Groups.* Vorlesungen aus dem Fachbereich Mathematik der Universität Essen, Heft 20, 1994.

[We95] A. J. Weir, Sylow p-Subgroups of the Classical Groups Over Finite Fields with Characteristic Prime to p. *Proc. Amer. Math. Soc.*, Vol. 6, No. 4 (Aug., 1995), 529–533.

Dipartimento di Matematica e Informatica, U. Dini, Viale Morgagni 67/a, Firenze, Italy.
E-mail address: eugenio.giannelli@unifi.it

Dept. Mathematical and Computer Sciences, MSU Denver, Denver, CO 80217, USA
E-mail address: aschaef6@msudenver.edu

ICMAT, Campus Cantoblanco UAM, C/ Nicolás Cabrera, 13-15, 28049 Madrid, Spain
E-mail address: carolina.vallejo@icmat.es